On the existence of LOCC-distinguishable bases in three-dimensional subspaces of bipartite 3 × *n* systems

2007 J. Phys. A: Math. Theor. 40 7939

(http://iopscience.iop.org/1751-8121/40/28/S04)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7939–7944 doi:10.1088/1751-8113/40/28/S04

On the existence of LOCC-distinguishable bases in three-dimensional subspaces of bipartite $3 \times n$ systems

Christopher King[1] **and Daniel Matysiak**[2]

[1] Department of Mathematics, Northeastern University, Boston MA 02115, USA
[2] College of Computer and Information Science, Northeastern University, Boston MA 02115, USA

Received 30 October 2006, in final form 9 February 2007
Published 27 June 2007
Online at stacks.iop.org/JPhysA/40/7939

Abstract

We present numerical evidence showing that any three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$ has an orthonormal basis which can be reliably distinguished using one-way LOCC (local operations and classical communication), where a measurement is made first on the three-dimensional part and the result used to select an optimal measurement on the n-dimensional part. We also show that the order of measurement is essential, by providing an example of a three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^5$ which does not have any basis that can be distinguished by measuring first on the five-dimensional factor.

PACS numbers: 03.65.Ud, 03.67.−a

1. Introduction

In a bipartite space $\mathcal{A} \otimes \mathcal{B}$, orthogonal entangled states can be reliably distinguished using entangled measurements. However it is generally not possible to reliably distinguish such states on $\mathcal{A} \otimes \mathcal{B}$ using product measurements, or even more general separable measurements. This question of distinguishing orthogonal entangled states using LOCC (local operations and classical communication) has been investigated in several recent papers in a variety of settings [1–7, 10].

In this paper we follow a slightly different line of enquiry, by asking whether a given subspace of the product $\mathcal{A} \otimes \mathcal{B}$ contains any orthonormal basis which can be reliably distinguished using LOCC. An early positive result in this direction is due to Walgate *et al* [12], who showed that any two orthogonal bipartite states can be reliably distinguished using LOCC. This implies in particular that every two-dimensional subspace of a bipartite space has a basis that can be reliably distinguished using LOCC. In the direction of negative results, Gregoratti and Werner [6] proved the existence of bipartite subspaces which do not have any basis that can be reliably distinguished using LOCC. Recently Watrous [13] has constructed explicit examples of $(d^2 - 1)$-dimensional subspaces in $\mathbb{C}^d \otimes \mathbb{C}^d$ which have no basis that can be distinguished using LOCC, for $d \geqslant 3$. In fact Watrous' result is even stronger, because he proves that there is no separable POVM which can distinguish any basis. In related work Winter [14] has found an expression for the optimal asymptotic rate of transferring bipartite quantum states to one party using LOCC, thereby obtaining a lower bound for the optimal asymptotic rate of distinguishing states using LOCC.

In this paper we report on numerical investigations of this question in some low-dimensional cases, namely for three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ with $n \geqslant 3$. We have found in every case that there is an orthonormal basis of the subspace which can be reliably distinguished using one-way LOCC, where measurements are made first on $\mathbb{C}^3$, and the result used to select the optimal measurement on $\mathbb{C}^n$. We have also found asymmetry in the order of measurements for $n \geqslant 5$—in general, it is not possible to find a basis which can be distinguished by first measuring on $\mathbb{C}^n$, and then using the result to select the optimal measurement on $\mathbb{C}^3$.

Our investigations are partially motivated by the idea of using measurements on the environment to reduce the noise in a quantum channel [7, 8, 14]. The Lindblad–Stinespring representation [9, 11] shows how the effect of noise on a quantum system can be understood as a two-step process: first the system and environment are entangled by a non-local unitary operator U, and then the environment is traced out. The idea here is to replace the second step by a measurement on the environment, and then to pass the result of this measurement to the system side, where it can be used to select the best measurement strategy there. In general this should allow more information to be reliably recovered from the channel.

To be more specific, if the system is in a pure state $|\psi\rangle$ and the environment is in a pure state $|\omega\rangle$, then the unitary operator maps the initial product state $|\psi\rangle \otimes |\omega\rangle$ to the entangled state $U(|\psi\rangle \otimes |\omega\rangle)$. As $|\psi\rangle$ varies over the system state space, the states $U(|\psi\rangle \otimes |\omega\rangle)$ vary over a subspace V of the system plus environment. Suppose that some set of states $\{U(|\psi_i\rangle \otimes |\omega\rangle)\}$ in V can be reliably distinguished using one-way LOCC. Then we can encode classical information in the system states $\{|\psi_i\rangle\}$, and completely recover this information at the output by using coordinated measurements of the environment and the system. If V has a basis of states with this property, then the classical information-carrying capacity of the channel is $\log d$, where d is the system dimension. In this setting our conjecture would imply that every channel with a three-dimensional environment has environment-assisted classical capacity equal to $\log 3$.

2. Results

Our results are summarized in the following statement. Because we rely on numerical investigations at this time, we pose the statement as a conjecture.

Conjecture 1. *Let V be a three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$, with $n \geqslant 3$. Then there is an orthonormal basis of V which can be reliably distinguished using one-way LOCC, where measurements are made first on $\mathbb{C}^3$, and the result used to select the optimal measurement on $\mathbb{C}^n$.*

2.1. Discussion of the numerical investigations

The numerical search was conducted by randomly selecting three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$, for the range of values $3 \leqslant n \leqslant 9$. In each subspace a search was performed over randomly selected orthonormal bases $\{|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle\}$ and over random partial measurements on the $\mathbb{C}^3$ part. The partial measurement on $\mathbb{C}^3$ was performed by projecting onto a randomly

Table 1. Numerical data for three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ with $n = 3, 4, \ldots, 9$, namely: the average value of the objective function H for values less than the threshold value of 10^{-6}, and the number of tested subspaces.

n	Average H	Number of subspaces
3	$2.816\,258 \times 10^{-8}$	138 211
4	$3.789\,893 \times 10^{-8}$	30 271
5	$3.789\,893 \times 10^{-8}$	32 278
6	$4.127\,063 \times 10^{-8}$	30 000
7	$4.394\,015 \times 10^{-8}$	30 000
8	$5.130\,496 \times 10^{-8}$	30 216
9	$5.594\,670 \times 10^{-8}$	30 006

selected orthonormal basis $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$. Denoting by $|\phi(i,a)\rangle$ $(i, a = 1, 2, 3)$ the projection of $|\theta_i\rangle$ onto $|v_a\rangle$, it follows that the condition for reliably distinguishing the basis states $\{|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle\}$ with this partial measurement is that for each measurement outcome a the three states $|\phi(1,a)\rangle, |\phi(2,a)\rangle, |\phi(3,a)\rangle$ should be orthogonal. Define

$$H(V) = \min \left\{ \sum_{a=1}^{3} \sum_{i \neq j=1}^{3} |\langle \phi(i,a)|\phi(j,a)\rangle|^2 \right\} \tag{1}$$

where the minimization is performed over all bases $|\theta_i\rangle$ and partial measurements $|v_a\rangle$. Clearly $H(V) \geqslant 0$, and $H(V) = 0$ if and only if for some basis and partial measurement the vectors $\{|\phi(i,a)\rangle\}_{i=1}^{3}$ are orthogonal for each outcome a.

The minimization problem described above was implemented using the package TOMLAB. Table 1 was obtained by randomly sampling three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$, and for each subspace searching for an orthonormal basis that could be reliably distinguished using one-way LOCC as described above. Column 3 shows the number of subspaces sampled for each value of n. For each subspace, the objective function (1) was determined by minimizing over choices of orthonormal basis in V and partial measurements in $\mathbb{C}^3$. The threshold value 10^{-6} was used to terminate the search for the minimum value. In every case this threshold was reached. To test robustness, the algorithm was run with different starting values for the same subspace, and in every case the same solution was discovered. The algorithm was also tested on cases where the solution could be determined by hand. Column 2 shows the average minimum value of the objective function when the search was terminated. As explained in the next section, it is sufficient to consider values of n in the range $3 \leqslant n \leqslant 9$.

3. Mathematical formulation

The mathematical statement of the result in the case $\mathbb{C}^3 \otimes \mathbb{C}^n$ is the following. From the definitions of $\theta_i\rangle$, $|v_a\rangle$ and $|\phi(i,a)\rangle$ it follows that

$$|\theta_i\rangle = \sum_{a=1}^{3} |v_a\rangle \otimes |\phi(i,a)\rangle \tag{2}$$

for $i = 1, 2, 3$. For each $a, b = 1, 2, 3$ define the 3×3 matrix M_{ab} by

$$(M_{ab})_{ij} = \langle \phi(i,a)|\phi(j,b)\rangle \tag{3}$$

and let M be the 9×9 positive semidefinite matrix whose 3×3 blocks are M_{ab}, that is

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}. \tag{4}$$

Then M is positive semidefinite, and

$$M_{11} + M_{22} + M_{33} = I. \tag{5}$$

Furthermore $M = R^*R$ where R denotes the $n \times 9$ matrix with columns $|\phi(i,a)\rangle$. The condition that the states $|\phi(i,a)\rangle$ are orthogonal for each measurement result is that the matrices M_{11}, M_{22}, M_{33} are diagonal.

Changing the basis vectors $|\theta_i\rangle$ in the subspace induces a map $M_{ab} \mapsto W M_{ab} W^*$ $(a, b = 1, 2, 3)$ for some element W of $SU(3)$. In a similar way, changing the measurement basis in $\mathbb{C}^3$ induces the map $M_{ab} \mapsto \sum_{c,d} U_{ac} M_{cd} U^*_{db}$ for some matrix U in $SU(3)$. Hence conjecture 1 implies that there are unitaries $W, U \in SU(3)$ such that the diagonal blocks of the conjugated matrix

$$(W \otimes U) M (W \otimes U)^* \tag{6}$$

commute.

Now suppose that M is any positive semidefinite 9×9 matrix satisfying (5). Then we can write $M = T^*T$ for some $k \times 9$ matrix T, with $k \leqslant 9$, and the columns of T provide a set of vectors which arise by performing partial measurements on some triplet of orthogonal states in $\mathbb{C}^3 \otimes \mathbb{C}^k$. It follows that every positive semidefinite matrix of the form (4) satisfying (5) can be associated with a three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^k$ with a chosen orthonormal basis, and with a specified basis of $\mathbb{C}^3$. So conjecture 1 can be reformulated as the statement that for any positive semidefinite 9×9 matrix satisfying (5), there are unitaries $W, U \in SU(3)$ such that the diagonal blocks of the conjugated matrix (6) commute.

Now consider the case of distinguishing a basis in a subspace V of $\mathbb{C}^3 \otimes \mathbb{C}^n$ with $n > 9$. As described above this leads to a positive semidefinite 9×9 matrix M satisfying (5). So if the conjecture holds, then there are $W, U \in SU(3)$ such that the diagonal blocks of the matrix (6) commute. Applying these transformations to the basis of V and the measurement basis in $\mathbb{C}^3$ generates a set of orthogonal states in $\mathbb{C}^n$ for each measurement outcome, which is the desired result for V. Hence it is sufficient to prove the conjectured commutativity result for the conjugated matrix (6), and therefore it is sufficient to show that distinguishability holds for subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ with $n \leqslant 9$.

4. Related results

4.1. One-way LOCC subspace discrimination is not symmetric

A numerical search readily turns up examples of three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ which have no basis that can be reliably distinguished using one-way LOCC with partial measurement first on the $\mathbb{C}^n$ factor, for $n \geqslant 5$. We present one of these examples below. Interestingly, three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^4$ do not seem to share this property.

The three states $\{|\theta_1\rangle, |\theta_2\rangle, |\theta_3\rangle\}$ below generate a subspace of $\mathbb{C}^3 \otimes \mathbb{C}^5$ which numerical evidence shows does not have a basis that can be reliably distinguished using one-way LOCC, with partial measurement first made on the $\mathbb{C}^5$ factor.

$$|\theta_1\rangle = \begin{pmatrix} -0.2450 - 0.0054i \\ -0.1694 + 0.0815i \\ 0.1071 - 0.3191i \\ 0.0655 - 0.3190i \\ -0.1911 - 0.1862i \\ 0.1185 + 0.3259i \\ -0.2530 + 0.0480i \\ 0.1194 - 0.1987i \\ 0.1948 - 0.2106i \\ 0.0595 + 0.2934i \\ 0.1286 - 0.1427i \\ -0.1420 + 0.1308i \\ -0.2367 + 0.1399i \\ 0.1384 - 0.0264i \\ 0.0867 + 0.1573i \end{pmatrix}, \quad |\theta_2\rangle = \begin{pmatrix} 0.1438 + 0.2108i \\ -0.3214 + 0.1308i \\ 0.1229 + 0.0319i \\ 0.1775 - 0.1070i \\ 0.2091 - 0.1811i \\ -0.0937 + 0.1880i \\ 0.1609 + 0.0272i \\ 0.1705 + 0.0996i \\ -0.0630 + 0.0729i \\ 0.3389 - 0.1242i \\ 0.0201 - 0.2668i \\ 0.1127 - 0.3331i \\ 0.2338 + 0.3325i \\ -0.1798 - 0.0796i \\ -0.1097 + 0.1360i \end{pmatrix}$$

$$|\theta_3\rangle = \begin{pmatrix} 0.0390 - 0.0484i \\ 0.0405 - 0.2603i \\ 0.2206 + 0.2432i \\ -0.2843 - 0.0751i \\ -0.2416 - 0.1380i \\ 0.0510 + 0.3270i \\ 0.1691 + 0.0829i \\ -0.3761 - 0.1033i \\ 0.0988 + 0.1388i \\ 0.3138 + 0.2228i \\ 0.0553 + 0.2272i \\ 0.0468 - 0.0164i \\ 0.1966 - 0.1044i \\ -0.0147 + 0.1239i \\ -0.2313 + 0.0715i \end{pmatrix}.$$

4.2. Channel capacity assisted by LOCC

In some cases it may be possible to more reliably distinguish output states of a noisy quantum channel by using measurements on the environment in addition to measurements on the system. This idea of using information from the environment to enhance channel capacity has been pursued in a number of settings [7, 8, 14]. Winter [14] introduced the notation *environment-assisted* to mean a measurement which is first performed on the environment, and where the result is used to select an optimal measurement on the system; and *environment-assisting*, meaning a measurement which is first performed on the system, and where the result is used to select an optimal measurement on the environment.

As discussed in the introduction, the Lindblad–Stinespring representation [9, 11] allows the question of finding the environment-assisted/assisting classical capacity to be formulated as the problem of finding a basis of an entangled subspace of the system plus environment which can be reliably distinguished using one-way LOCC. In this setting our conjecture would imply that every channel with a three-dimensional environment has environment-assisted classical capacity equal to $\log 3$, and every qutrit channel has environment-assisting classical capacity equal to $\log 3$.

5. Conclusions

We have used numerical techniques to investigate the effectiveness of one-way LOCC in reliably distinguishing some basis of a subspace in a bipartite space. Our results show that every three-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^n$ has a basis which can be distinguished by one-way LOCC, when measurements are first performed on the $\mathbb{C}^3$ part and the result is then used to select the optimal measurement on the $\mathbb{C}^n$ part. The same numerical techniques show that one-way LOCC is not symmetric, and that many three-dimensional subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^n$ do not have any basis that can be distinguished using one-way LOCC starting with a measurement on the $\mathbb{C}^n$ part, for $n \geqslant 5$.

There are several further directions to pursue in this line of research. One direction is to look for an analytical proof of our results, possibly by extending work of Nathanson [10] and others on LOCC state discrimination. Another direction is to continue numerical investigations in higher dimensions. We plan to work along both of these lines of investigation.

Acknowledgments

This research was supported in part by National Science Foundation Grant DMS-0400426. The authors are grateful to Javed Aslam for use of computing facilities.

References

[1] Bennett C, DiVincenzo D, Fuchs C, Mor T, Rains E, Shor P, Smolin J and Wootters W 1999 Quantum nonlocality without entanglement *Phys. Rev.* A **59** 1070
[2] Chen P-X and Li C-Z 2003 Orthogonality and distinguishability: criterion for local distinguishability of arbitrary orthogonal states *Phys. Rev.* A **68** 062107
[3] Fan H 2004 Distinguishability and indistinguishability by LOCC *Phys. Rev. Lett.* **92** 177905
[4] Ghosh S, Kar G, Roy A and Sarkar D 2004 Distinguishability of maximally entangled states *Phys. Rev.* A **70** 022304
[5] Ghosh S, Kar G, Roy A A, Sen De and Sen U 2001 Distinguishability of the Bell states *Phys. Rev. Lett.* **87** 277902
[6] Gregoratti M and Werner R F 2002 Quantum lost and found *J. Mod. Opt.* **50** 915
[7] Gregoratti M and Werner R F 2004 On quantum error correction by classical feedback in discrete time *J. Math. Phys.* **45** 2600
[8] Hayden P and King C 2005 Correcting quantum channels by measuring the environment *Quantum Inform. Comput.* **5** 156–60
[9] Lindblad G 1975 Completely positive maps and entropy inequalities *Commun. Math. Phys.* **40** 147–51
[10] Nathanson M 2005 Distinguishing bipartite orthogonal states using LOCC: best and worst cases *J. Math. Phys.* **46** 062103
[11] Stinespring W F 1955 Positive functions on C^*-algebras *Proc. Am. Math. Soc.* **6** 211–6
[12] Walgate J, Short A J, Hardy L and Vedral V 2000 Local distinguishability of multipartite orthogonal quantum states *Phys. Rev. Lett.* **85** 4972
[13] Watrous J 2005 Bipartite subspaces having no bases distinguishable by local operations and classical communication *Phys. Rev. Lett.* **95** 080505
[14] Winter A 2007 On environment-assisted capacities of quantum channels *Preprint* quant-ph/0507045 *Markov Process. Relat. Fields* at press (J T Lewis memorial issue)